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INVESTIGATION OF THE EIGENVALUES OF A MATRIX OF INTEGRAL OPERATORS

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Abstract: This paper investigates the spectral properties and eigenvalues of a matrix consisting of integral operators. Matrix-valued integral operators arise naturally in functional analysis, mathematical physics, differential and integral equations, and related mathematical models. The main purpose of the study is to analyze the eigenvalue problem for a matrix of integral operators and to establish the conditions under which nontrivial eigenfunctions exist. Particular attention is paid to the structure of the operator matrix, the associated system of integral equations, and the compactness properties of its components.

Keywords: integral operator, matrix, eigenvalue, eigenfunction, spectrum, Hilbert space, compact operator, integral equation.

Introduction

Integral operators play an important role in modern functional analysis and in the mathematical description of various applied problems. They are widely used in the theory of integral equations, mathematical physics, differential equations, and quantum models. The spectral analysis of integral operators is especially important because eigenvalues and eigenfunctions characterize essential properties of the corresponding mathematical system.

An important class of such operators is formed by matrices of integral operators. These operators generalize scalar integral operators and allow systems with several interacting components to be studied within a unified framework. In this case, the unknown function is vector-valued and the spectral problem leads to a system of coupled integral equations.
The purpose of this paper is to investigate the eigenvalue problem associated with a matrix of integral operators acting in a suitable Hilbert space.

Literature Review

The spectral theory of integral operators has been extensively studied within functional analysis. Classical results concerning eigenvalues and characteristic values of integral equations were established by Hille and Tamarkin, whose work provided an important theoretical foundation for the spectral analysis of linear integral equations.

Pietsch investigated the eigenvalues of integral operators and developed results concerning their spectral properties and distribution. His studies, published in Mathematische Annalen, remain relevant to the analysis of eigenvalue distributions for integral operators.

König studied the eigenvalue distribution of compact operators and its applications to integral operators. His research shows that the decay and distribution of eigenvalues are closely related to the properties of the corresponding operators and their kernels.

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Hirschman investigated eigenvalues of certain integral operators and their asymptotic behavior. His work also established connections between eigenvalue distribution problems for integral operators and related finite-dimensional operator problems.

Let $H=L^2(a,b)$ be a Hilbert space, and consider the product space

$$= H \oplus H$$

A matrix of integral operators can be represented in the following form:

$$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are integral operators.

For the vector:

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

the action of the operator is defined as

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} A_{11}f_1 + A_{12}f_2 \\ A_{21}f_1 + A_{22}f_2 \end{pmatrix}$$

Thus, the matrix operator generates a coupled system in which the diagonal operators describe the action within each component, while the off-diagonal operators describe the interaction between the components.

The eigenvalue problem for the matrix of integral operators is formulated as

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where λ denotes an eigenvalue and $F \neq 0$ is the corresponding eigenfunction. Equivalently, the problem can be written as a system of two operator equations:

$A_{11}f_1 + A_{12}f_2 = \lambda f_1$

$A_{21}f_1 + A_{22}f_2 = \lambda f_2$

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This representation shows that the spectral problem for the matrix operator is equivalent to a coupled system of integral equations. The existence of a nonzero solution determines the eigenvalues of the operator matrix. Suppose that each operator A_{ij} is defined by an integral kernel $K_{ij}(x,y)$. Then

$$(A_{ij}f)(x) = \int_a^b K_{ij}(x,y)\, f(y)\, dy$$

Therefore, the eigenvalue problem takes the form of a system of coupled integral equations. The properties of the kernels K_{ij} determine important spectral characteristics of the matrix operator.

If the kernels K_{ij} belong to $L^2((a,b)\times(a,b))$, then the corresponding integral operators are Hilbert–Schmidt and, consequently, compact. Under these assumptions, the matrix operator is compact on $H \oplus H$. Hence, every nonzero spectral point is an eigenvalue of finite multiplicity, and the only possible accumulation point of the nonzero eigenvalues is zero.

If the matrix operator is self-adjoint, its eigenvalues are real. Moreover, eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the inner product of the product Hilbert space.

Results and Discussion

The analysis demonstrates that the spectral properties of a matrix of integral operators depend essentially on the properties of its integral kernels and on the interaction between the matrix components. The diagonal operators A_{11} and A_{22} determine the individual component behavior, whereas A_{12} and A_{21} describe the coupling between the components.

For symmetric operator matrices, appropriate relations between the entries ensure self-adjointness. In this case, the eigenvalues are real and the spectral decomposition can be studied using the theory of compact self-adjoint operators.

The obtained framework can also be used for numerical investigations. By approximating the kernels and replacing the integral operators by finite-dimensional matrices, one can obtain numerical approximations of the eigenvalues and study their dependence on the parameters of the kernels.

Conclusion

This paper investigated the eigenvalue problem for a matrix of integral operators acting in a Hilbert space. The matrix operator was represented as a coupled system of integral operators, and the corresponding eigenvalue problem was formulated in operator form. Under suitable assumptions on the integral kernels, the components of the operator matrix are compact.

Consequently, the nonzero eigenvalues are isolated and have finite multiplicity, while zero may be the only accumulation point of the spectrum. For self-adjoint operator matrices, the eigenvalues are real and the corresponding eigenfunctions have the standard orthogonality properties.

The presented approach provides a theoretical framework for further investigation of matrix-valued integral operators, coupled integral equations, and related spectral problems.

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