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A MIXED PROBLEM FOR AN EQUATION COMPOSED OF COMBINATIONS OF AKNS(+1) AND AKNS(-1) EQUATIONS IN THE CLASS OF PERIODIC INFINITE-GAP FUNCTIONS

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1. Introduction. Building on the ideas developed in [1–3], the authors of [4, 5] established the solvability of the initial-boundary value problem for the nonlinear negative-order modified Korteweg–de Vries–Liouville (nmKdV–L) equation of the form

$$\begin{cases} u_{xt} - e^{2u} - a(t)(u_{xxxt} - (2u_x\mu_{xt})_x) = 0, \\ \mu_{xx} = u_x^2, \end{cases}$$

in the class of infinite-gap functions that are periodic with respect to the spatial variable.

2. Formulation of the Problem. In this paper, we consider the initial-boundary value problem for a nonlinear system obtained by combining the AKNS(+1) and AKNS(−1) equations [6], given by

$$\begin{cases} p_{xt} = 2p\mu_t + a(t)\left[p_{xx} - 2p(p^2+q^2)\right] + b(t)p, \\ q_{xt} = 2q\mu_t + a(t)\left[q_{xx} - 2q(p^2+q^2)\right] + b(t)q, \\ \mu_x = p^2 + q^2, p = p(x,t), q = q(x,t), \mu = \mu(x,t), x \in \mathbb{R}, t \geq 0. \end{cases} \quad (1)$$

with conditions

$$\begin{cases} p(x,t)|_{t=0} = p_0(x), \;\; p_0(x+\pi) = p_0(x), \;\; x \in \mathbb{R}, \\ q(x,t)|_{t=0} = q_0(x), \;\; q_0(x+\pi) = q_0(x), \;\; x \in \mathbb{R}, \\ \mu(x,t)|_{x=0} = \mu_0(t), \;\; \mu_0(t) \in C^1(t>0) \cap C(t \geq 0), \\ [q_t(x,t) + \mu_t(x,t)]|_{x=0} = \alpha(t), \;\; \alpha(t) \in C(t \geq 0), \end{cases} \quad (2)$$

in the class of real-valued, infinite-gap functions that are π-periodic with respect to the spatial variable x and satisfy the following periodicity and regularity conditions:

$$\begin{cases} p(x+\pi,t) = p(x,t), \;\; q(x+\pi,t) = q(x,t), x \in \mathbb{R}, t > 0 \\ p(x,t), q(x,t) \in C_{x,t}^{2,1}(t>0) \cap C(t \geq 0), \;\; \mu(x,t) \in C_{x,t}^{1,1}(t>0) \cap C(t \geq 0). \end{cases} \quad (3)$$

Here, $a(t), b(t), \alpha(t) \in C\big([0;\infty)\big)$ are given continuous and bounded functions.

The main objective of this paper is to investigate the solvability of the mixed problem (1)–(3). More precisely, we aim to establish the minimal regularity assumptions on the initial data $p_0(x), q_0(x), \mu_0(t)$ and $\alpha(t)$ that guarantee the existence and uniqueness of a global solution to this problem.

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In this paper, we develop an algorithm for constructing periodic infinite-gap solutions $\big(p(x,t), q(x,t), \mu(x,t)\big), x \in \mathbb{R}, t > 0$, of the mixed problem (1)–(3) by reducing it to the inverse spectral problem for the periodic Dirac operator:

$$\mathcal{L}(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, \quad x,\tau \in \mathbb{R}, t > 0, \tag{4}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x,t) = \begin{pmatrix} p(x,t) & q(x,t) \\ q(x,t) & -p(x,t) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We note that the inverse spectral problem for the periodic Dirac operator was studied in [7].

Let $c(x,\lambda,\tau,t) = \big(c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t)\big)^T$ и $s(x,\lambda,\tau,t) = \big(s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t)\big)^T$ denote the solutions of equation (4) satisfying the initial conditions $c(0,\lambda,\tau,t) = (1,0)^T$ and $s(0,\lambda,\tau,t) = (0,1)^T$, respectively. The function $\equiv \Delta(\lambda,\tau,t) = c_1(\pi,\lambda,\tau,t) + s_2(\pi,\lambda,\tau,t)$ is called the Lyapunov function (or Floquet discriminant) associated with equation (4).

The intervals $(\lambda_{2n-1}, \lambda_{2n}), n \in \mathbb{Z}$, are called spectral gaps, where λ_n are the roots of the equations $\Delta(\lambda) \mp 2 = 0$. These roots coincide with the eigenvalues of the periodic or antiperiodic boundary value problems for equation (4), subject to the boundary conditions $\big(y(0) = \pm\, y(\pi)\big)$.

Let $\xi_n(\tau,t), n \in \mathbb{Z}$, denote the roots of the equation

$$s_1(\pi,\lambda,\tau,t) = 0.$$

These roots coincide with the eigenvalues of the Dirichlet boundary value problem for system (4), subject to the boundary conditions

$$y_1(0,\lambda,\tau,t) = 0, y_1(\pi,\lambda,\tau,t) = 0$$

and satisfy

$$\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}], n \in \mathbb{Z}.$$

The quantities $\xi_n(\tau,t), n \in \mathbb{Z}$,, together with the signs $\sigma_n(\tau,t) = sgn\{s_2(\pi,\xi_n,\tau,t) - c_1(\pi,\xi_n,\tau,t)\}$, $n \in \mathbb{Z}$ are called the spectral parameters of the operator $\mathcal{L}(\tau,t)$. The spectral parameters together with the spectral band edges constitute the spectral data of the Dirac operator $\mathcal{L}(\tau,t)$.

The problem of reconstructing the coefficient matrix $\Omega(x,t)$ of the operator $\mathcal{L}(\tau,t)$ from its spectral data is called the inverse spectral problem. The coefficients $p(x+\tau,t)$ and $\mathrm{q}(x+\tau,t)$ of the operator $\mathcal{L}(\tau,t)$ are uniquely determined by the spectral data

$$\{\lambda_n(\tau,t), \xi_n(\tau,t), \sigma_n(\tau,t) = \pm\, 1, n \in \mathbb{Z}\}.$$

Using the initial functions $p_0(x+\tau)$ and $q_0(x+\tau), \tau \in \mathbb{R}$, we construct the corresponding Dirac operator $\mathcal{L}(\tau,0)$ of the form

$$\mathcal{L}(\tau,0)y \equiv B\frac{dy}{dx} + \Omega_0(x+\tau)y = \lambda y, \quad x \in \mathbb{R}, \quad \tau \in \mathbb{R}.$$

The spectral band edges of the resulting operator are independent of the shift parameter τ, that is, $\lambda_n(\tau) = \lambda_n, n \in \mathbb{Z}$, whereas the spectral parameters $\xi_n^0(\tau)\ and\ \sigma_n^0(\tau)$ depend on τ and are periodic with respect to this parameter.

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By solving the direct spectral problem, we obtain the spectral data of the operator $\mathcal{L}(\tau, 0)$.

3. Results and Discussion

The main result of this paper is stated in the following theorem.

Theorem 1. Suppose that the initial data $p_0(x)$, $q_0(x)$, $\mu_0(t)$ and $\alpha(t)$ satisfy the conditions

$$p_0(x+\pi) = p_0(x) \in C^3(\mathbb{R}), \quad q_0(x+\pi) = q_0(x) \in C^3(\mathbb{R}),$$
$$\mu_0(t) \in C^1(t>0) \cap C(t \geq 0), \quad \alpha(t) \in C(t \geq 0),$$

then the mixed problem (1)–(3) admits uniquely determined infinite-gap solutions $p(x,t), q(x,t), \mu(x,t), x \in \mathbb{R}, t > 0$, which are represented by the following uniformly convergent series, respectively:

$$p(\tau,t) = \sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau,t) \right), \tag{5}$$

$$q(\tau,t) = \sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k(\tau,t) h_k\big(\xi(\tau,t)\big), \tag{6}$$

$$\mu(\tau,t) = \mu_0(t) + \int_0^\tau \left\{ \left(\sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau,t) \right) \right)^2 + \right.$$
$$\left. + \left(\sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k(\tau,t) h_k\big(\xi(\tau,t)\big) \right)^2 \right\} ds, \tag{7}$$

where

$$h_n\big(\xi(\tau,t)\big) = \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})\big(\lambda_{2n} - \xi_n(\tau,t)\big)} \cdot f_n\big(\xi(\tau,t)\big),$$

$$f_n\big(\xi(\tau,t)\big) = \sqrt{\prod_{\substack{k=-\infty, \\ k \neq n}}^{+\infty} \frac{\big(\lambda_{2k-1} - \xi_n(\tau,t)\big)\big(\lambda_{2k} - \xi_n(\tau,t)\big)}{\big(\xi_k(\tau,t) - \xi_n(\tau,t)\big)^2}}, \quad n \in \mathbb{Z}.$$

Moreover, the coordinates

$$\xi \equiv \xi(\tau,t) = (\ldots, \xi_{-n}(\tau,t), \ldots \xi_{-1}(\tau,t), \xi_0(\tau,t), \xi_1(\tau,t), \ldots, \xi_n(\tau,t), \ldots),$$
$$\sigma \equiv \sigma(\tau,t) = (\ldots, \sigma_n(\tau,t), \ldots, \sigma_{-1}(\tau,t), \sigma_0(\tau,t), \sigma_1(\tau,t), \ldots, \sigma_n(\tau,t), \ldots),$$

of the corresponding vectors satisfy the system of Dubrovin differential equations:

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^n \sigma_n(\tau,t) \times$$
$$\times \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})\big(\lambda_{2n} - \xi_n(\tau,t)\big)} f_n\big(\xi(\tau,t)\big) g_n\big(\xi(\tau,t)\big), n \in \mathbb{Z}, \tag{8}$$

with initial conditinos

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$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau),\ \ \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \in \mathbb{Z}. \tag{9}$$

Here

$$g_n\big(\xi(\tau,t)\big) = -\,a(t)\left[p(\tau,t) + \xi_n(\tau,t) + \frac{1}{2\xi_n(\tau,t)}\big(p^2(\tau,t) + q^2(\tau,t) + q_\tau(\tau,t)\big)\right] +$$

$$+\frac{b(t)}{4\xi_n(\tau,t)} + \frac{q_t(\tau,t) + \mu_t(\tau,t)}{2\xi_n(\tau,t)}.$$

By introducing the change of variables

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1})\sin^2 x_n(\tau,t)\,,\ \ n \in \mathbb{Z},$$

the system of Dubrovin differential equations (8), together with the initial conditions (9), can be reformulated as a single differential equation in the Banach space $\mathbb{K}$:

$$\frac{dx(\tau,t)}{dt} = H\big(x(\tau,t)\big),\ \ x(\tau,t)|_{t=0} = x^0(\tau) \in \mathbb{K}, \tag{10}$$

where

$$\mathbb{K} = \left\{x = (\ldots, x_{-1}(\tau,t), x_0(\tau,t), x_1(\tau,t), \ldots) : \|x\| = \sum_{n=-\infty}^{+\infty} (\lambda_{2n} - \lambda_{2n-1})|x_n| < \infty\right\},$$

$$H(x) = (\ldots, H_{-1}(x), H_0(x), H_1(x), \ldots),$$

$$H_n(x) = (-1)^n \sigma_n^0(\tau) f_n(\ldots, \lambda_{-1} + (\lambda_0 - \lambda_{-1})\sin^2 x_0(\tau,t), \ldots) \times$$

$$\times\, g_n(\ldots, \lambda_{-1} + (\lambda_0 - \lambda_{-1})\sin^2 x_0(\tau,t), \ldots) = (-1)^n \sigma_n^0(\tau)\bar{f}_n\big(x(\tau,t)\big)\bar{g}_n\big(x(\tau,t)\big),$$

$$\sigma_n^0(\tau) = \sigma_n(\tau,t)sgn\{\sin x_n(\tau,t)\cos x_n\,(\tau,t)\}.$$

Under the assumptions $p_0(x+\pi) = p_0(x) \in C^3(\mathbb{R})$, $q_0(x+\pi) = q_0(x) \in C^3(\mathbb{R})$ and using the asymptotic behavior of the eigenvalues of the periodic and antiperiodic boundary value problems for the Dirac system, we obtain the following relations for the spectral gap lengths (see [6]):

$$\begin{cases} \gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \dfrac{|q_{2k}^3|}{2^2|k|^3} + \dfrac{\delta_k}{|k|^4}, \\ \displaystyle\sum_{k=-\infty}^{\infty} |q_{2k}^3|^2 < \infty, \ \sum_{k=-\infty}^{\infty} \delta_k^2 < \infty. \end{cases}$$

Lemma 1. The following inequalities hold:

$$C_1 \le |\bar{f}_n(x)| \le C_2, \ \ \left|\frac{\partial \bar{f}_n(x)}{\partial x_m}\right| \le C_3\gamma_m,$$

$$|\bar{g}_n(x)| \le C_4(1 + |n|), \ \ \left|\frac{\partial \bar{g}_n(x)}{\partial x_m}\right| \le C_5\gamma_m(1 + |m|),$$

where $C_j > 0, j = 1,2,3,4,5,$ do not depend on the parameters m and n, also $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$.

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Lemma 2. Suppose that the initial functions \(p_0(x)\) and \(q_0(x)\) satisfy the conditions

$$p_0(x+\pi) = p_0(x) \in C^3(\mathbb{R}), \;\; p_0(x+\pi) = p_0(x) \in C^3(\mathbb{R}),$$

then the Cauchy problem (10) admits a unique global solution that is stable.

Remark 1. Theorem 1 and Lemma 2 provide an algorithm for constructing infinite-gap solutions of the mixed problem (1)–(3) using the inverse spectral method. The procedure consists of the following steps:

1. First, we determine the spectral data $\{\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}\}$ of the Dirac operator $\mathcal{L}(\tau, 0)$ by solving the corresponding direct spectral problem.

2. Next, using these spectral data as initial conditions, we solve the Cauchy problem (8)–(9) for an arbitrary value of τ, thereby obtaining the time-dependent spectral parameters $\xi_n(\tau, t), \sigma_n(\tau, t), n \in \mathbb{Z}$.

3. Finally, applying the reconstruction formulas (5)–(7), we recover the functions $p(\tau, t), q(\tau, t)$ and $\mu(\tau, t)$, which constitute a solution of the mixed problem (1)–(3).

4. Conclusion

The inverse spectral method is applied to integrate a nonlinear system obtained by combining the AKNS(+1) and AKNS(−1) equations in the class of periodic infinite-gap functions. The existence and uniqueness of solutions to the initial value problem for the associated infinite system of Dubrovin differential equations are established in the class of three-times continuously differentiable periodic infinite-gap functions. Furthermore, the solvability of the Cauchy problem for the combined AKNS(+1)–AKNS(−1) system is established within the same class of functions.

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